J. Fluid Mech. (1990), *vol.* 213, *pp.* 191–201
Printed in Great Britain

Non-axisymmetric viscous lower-branch modes in axisymmetric supersonic flows

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(Received 2 August 1988 and in revised form 9 May 1989)

In a previous paper, we have considered the weakly nonlinear interaction of a pair of axisymmetric lower branch Tollmien–Schlichting instabilities in cylindrical supersonic flows. Here the possibility that non-axisymmetric modes might also exist is investigated. In fact it is found that such modes do exist and, on the basis of linear theory, it appears that these modes are the most important. The non-axisymmetric modes are found to exist for flows around cylinders with non-dimensional radius a less than some critical value a_c. This critical value a_c is found to increase monotonically with the azimuthal wavenumber n of the disturbance and it is found that unstable modes always occur in pairs. We show that, in general, instability in the form of lower branch Tollmien–Schlichting waves will occur first for non-axisymmetric modes and that in the unstable regime, the largest growth rates correspond to the latter modes.

1. Introduction

Recent interest in the development of supersonic and hypersonic vehicles has stimulated research into the different instability mechanisms which might cause transition to turbulence in high-speed compressible flows. Here an investigation of the role of non-axisymmetric lower-branch Tollmien–Schlichting waves in supersonic flows with an axis of symmetry is investigated. Such a study is crucial for verifying the feasibility of these vehicles, since experimental investigations in these regimes are extremely expensive and difficult to carry out.

Following the work of Smith (1979*a*, *b*) it is now well known that triple-deck theory provides a self-consistent asymptotic framework for the description of lower-branch linear and nonlinear, two- and three-dimensional Tollmien–Schlichting waves. Thus it is known that finite-amplitude Tollmien–Schlichting waves are stabilized by nonlinear effects as they cross the lower branch of the neutral curve. At higher disturbance amplitudes further downstream Smith & Burggraf (1985) have shown that a hierarchy of fully nonlinear states can be achieved. The planar compressible problem has been investigated using triple-deck theory by Smith (1989). At subsonic speeds lower-branch disturbances are essentially unchanged from their incompressible forms. However, at supersonic speeds only three-dimensional modes can be unstable and the critical angle of propagation above which instability can occur increases with the Mach number. At hypersonic speeds Smith (1989) has shown that the modes are then fully non-parallel and a quasi-parallel theory based on triple-deck theory fails. It can be shown that in this regime the modes have a structure similar to that which describes Görtler vortices in incompressible flows (see for example Hall 1983).

In a previous paper, Duck & Hall (1989), we investigated the linear and weakly nonlinear theory of lower-branch axisymmetric Tollmien–Schlichting instabilities in supersonic cylindrical flows. We found that such modes exist in pairs and that at a given Mach number they occur only for a body radius less than a critical value. In the limit of small body radius both modes have wavelength going to zero with respect to the usual triple-deck streamwise lengthscale. In the weakly nonlinear stage it was shown that, dependent on the input frequency, either mode can lead to a stable finite-amplitude state.

Here we generalize the above calculation to see if non-axisymmetric modes can be important in the linear regime. We perturb an axisymmetric supersonic flow to three-dimensional disturbances with wavenumbers α and n in the streamwise and azimuthal directions. We show that at all supersonic speeds there is a finite band of non-dimensional body radii that can support two three-dimensional modes of the same azimuthal wavenumber, and that in the limit of scaled body radius tending to zero one mode has wavelength tending to zero whilst the other tends to infinity. Thus for these cylinders Tollmien–Schlichting waves with wavelengths both short and long compared with the usual streamwise triple-deck scale can occur. The planar problem has been recently studied by Smith (1989) who showed that three-dimensional disturbances are the most unstable over a wide range of Mach numbers. Indeed in the hypersonic limit Smith showed that they were the only possible type of unstable Tollmien–Schlichting wave. We shall show that, on the basis of linear theory, it is the three-dimensional modes that are also the most dangerous in cylindrical flows since they occur at the lowest Reynolds number and when they occur have the largest growth rates.

The procedure adopted in the rest of this paper is as follows: in §2 we formulate the linear stability problem for supersonic cylindrical flows. In §3 we determine the eigenrelations for three-dimensional Tollmien–Schlichting waves around a cylinder with radius comparable with the upper-deck thickness. In §4 the eigenrelation is investigated in the limit of small and large cylinders and also in the high azimuthal wavenumber limit. Finally in §5 we describe some results and draw some conclusions.

2. Formulation

We shall be concerned with the linear stability of an axisymmetric boundary layer on a cylindrical body of radius a^*, in a uniform supersonic stream of velocity U^*_∞ aligned with the axis of the cylinder.

If L^* denotes a typical streamwise lengthscale (for example the distance from some leading edge), ν^*_∞ the kinematic viscosity of the fluid in the far field, then the Reynolds number Re is defined to be

$$Re = U^*_\infty L^*/\nu^*_\infty. \tag{2.1}$$

It is found useful to introduce a related parameter ϵ,

$$\epsilon = Re^{-\frac{1}{8}}. \tag{2.2}$$

In this paper the Reynolds number is taken to be large, implying ϵ is small.

It is also assumed that

$$\bar{a} = \frac{a^*}{\epsilon^3 L^*} = O(1) \tag{2.3}$$

denotes the scale of the radius of the body. This follows the scale chosen by Duck & Hall (1989), Kluwick, Gittler & Bodonyi (1984), and one of the scales chosen by Duck (1984), amongst others; this turns out to be an important choice of body radius scale, with curvature terms playing a crucial role in the physics of the problem.

The problem then takes on a triple-deck structure. The following non-dimensional variables are defined

$$\left.\begin{aligned} \bar{X} &= \frac{x^*-L^*}{\epsilon^3 L^*}, \quad \bar{r} = \frac{r^*}{\epsilon^3 L^*}, \quad \bar{p} = \frac{p^*-p^*_\infty}{\rho^*_\infty U^{*2}_\infty}, \quad \bar{u} = \frac{u^*}{U^*_\infty}, \\ \bar{v} &= \frac{v^*}{U^*_\infty}, \quad \bar{w} = \frac{w^*}{U^*_\infty}, \quad c = \frac{c^*}{U^*_\infty}, \quad \bar{t} = \frac{U^*_\infty t^*}{\epsilon^2 L^*}. \end{aligned}\right\} \tag{2.4}$$

Here x^*, r^* and θ are taken to be the streamwise, radial and azimuthal coordinates respectively at some suitable reference point on the body, and u^*, v^*, w^* are the corresponding velocity components, c^* denotes the speed of sound, ρ^*_∞ the fluid density far from the body, p^* the pressure, p^*_∞ the far-field pressure, and t^* the time.

We confine our study to the stability of the flow at a location on the body where the boundary-layer thickness is $O(\epsilon^4 L^*)$, which is thin compared with the radius of the body. The skin friction of the undisturbed boundary layer is then taken to be $U^*\lambda\epsilon^4/L^*$, where λ is some order-one parameter. The flow is taken to be parallel, a completely rational approximation in this context because of the small, $O(\epsilon^3 L^*)$, streamwise lengthscale under consideration (although non-parallel effects could become important in a nonlinear study).

We consider first the upper deck of the triple deck, where $\bar{r} = O(1)$. Here the perturbation pressure and sound speed expand as

$$p = \epsilon^2 p_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \ldots, \quad c = M_\infty^{-1} + \epsilon^2 c_1(\bar{X}, \bar{r}, \theta, \bar{t}) + \ldots. \tag{2.5}$$

M_∞ is the Mach number of the external flow. The flow in this layer turns out to be completely irrotational, and may be reduced to the Prandtl–Glauert equation (expressed in cylindrical polar coordinates) for the pressure p_1,

$$(1-M_\infty^2)\, p_{1\bar{x}\bar{x}} + \frac{1}{\bar{r}} p_{1\bar{r}} + p_{1\bar{r}\bar{r}} + \frac{1}{\bar{r}^2} p_{1\theta\theta} = 0. \tag{2.6}$$

Here subscripts with respect to variables denote partial differentiation. The solution of this will be deferred until the appropriate boundary conditions have been ascertained. We consider next the main and lower decks, which correspond to the transverse scale

$$y = \frac{r^*-a^*}{\epsilon^4 L^*} = O(1), \quad Y = \frac{y}{\epsilon} = O(1). \tag{2.7}$$

The appropriate expansions in these layers take the form

$$(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \rho) = \left(U_0 + \epsilon \bar{A} U_0', \, -\epsilon^2 \bar{A}_{\bar{X}} U_0, \frac{\epsilon^2 \bar{D}}{U_0}, \epsilon^2 \bar{P}, R_0 + \epsilon \bar{A} R_0'\right) + \ldots, \tag{2.8a}$$

$$(\bar{u}, \bar{v}, \bar{w}, \bar{p}, \rho) = \left(\epsilon \bar{U}, \epsilon^3 \bar{U}, \epsilon \bar{W}, \epsilon^2 \bar{P}, R_0(0) + \epsilon \rho_1\right) + \ldots. \tag{2.8b}$$

Here the displacement function $\bar{A}$ and $\bar{P}$ depend on $\bar{X}, \theta, \bar{t}$, whilst $\bar{U}, \bar{V}, \bar{W}, \bar{P}$ and ρ depend on $\bar{X}, \theta, \bar{t}$ and $\bar{Y}$. In these decks curvature effects are negligible at the level of approximation discussed here. The function $\bar{D}$ is in fact equal to $-\bar{P}_\theta/\bar{a}$ and the usual matching condition between the main and upper decks yields $P_1|_{\bar{r}=\bar{a}} \bar{A}_{\bar{x}\bar{x}}$. It is now possible to scale out a number of the physical constants. Following Kluwick *et al.* (1984) and Duck & Hall (1989), this is achieved as follows:

$$\left.\begin{aligned}&\bar{X} = C^{\frac{3}{8}}\lambda^{-\frac{5}{4}}(T_w/T_\infty)^{\frac{3}{2}}X, \quad \bar{Y} = C^{\frac{5}{8}}\lambda^{-\frac{3}{4}}(T_w/T_\infty)^{\frac{3}{2}}Y, \quad \bar{P} = C^{\frac{1}{4}}\lambda^{\frac{1}{2}}P,\\ &\bar{U} = C^{\frac{1}{8}}\lambda^{\frac{1}{4}}(T_w/T_\infty)^{\frac{1}{2}}U, \quad \bar{V} = C^{\frac{3}{8}}\lambda^{\frac{3}{4}}(T_w/T_\infty)^{\frac{1}{2}}V, \quad \bar{W} = C^{\frac{1}{8}}\lambda^{\frac{1}{4}}(T_w/T_\infty)^{\frac{3}{2}}W,\\ &\bar{A} = C^{\frac{3}{8}}\lambda^{-\frac{3}{4}}(T_w/T_\infty)^{\frac{3}{2}}A, \quad \bar{a} = C^{\frac{3}{8}}\lambda^{-\frac{5}{4}}(T_w/T_\infty)^{\frac{3}{2}}a, \quad \bar{t} = C^{\frac{1}{4}}\lambda^{-\frac{3}{2}}(T_w/T_\infty)\,t.\end{aligned}\right\} \tag{2.9}$$

Here C is the Chapman constant arising from a linear viscosity law and T_w is the non-dimensional wall temperature. The governing equations in the lower deck then turn out to be

$$\left.\begin{aligned}&\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\frac{W}{a}\frac{\partial U}{\partial \theta}=\frac{\partial^2 U}{\partial Y^2}-\frac{\partial P}{\partial X},\\ &\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}+\frac{W}{a}\frac{\partial W}{\partial \theta}=\frac{\partial^2 W}{\partial Y^2}-\frac{1}{a}\frac{\partial P}{\partial \theta},\\ &\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}+\frac{1}{a}\frac{\partial V}{\partial \theta}=0,\\ &U=V=W=0 \quad \text{on} \quad Y=0,\\ &U\rightarrow Y+A(X,\theta,t) \quad \text{as} \quad Y\rightarrow\infty,\\ &\qquad W\rightarrow D/Y \quad \text{as} \quad Y\rightarrow\infty,\end{aligned}\right\} \tag{2.10}$$

If we write the solution of (2.6) symbolically as

$$P = \mathscr{L}\{A\} \tag{2.11}$$

then the problem is effectively closed.

3. Linear stability

The lower deck is then linearized about the basic state by writing, for example, A in the form $A = [\hat{A}_1 \exp \mathrm{i}(\alpha X + n\theta - \Omega t) + \text{complex conjugate}]$ so that the continuity and momentum equations reduce to:

$$\hat{U}_1(-\mathrm{i}\Omega+\mathrm{i}\alpha Y)+\hat{V}_1 = \mathrm{i}\alpha\hat{P}_1+\hat{U}_{1YY}, \tag{3.1a}$$

$$\hat{W}_1(-\mathrm{i}\Omega+\mathrm{i}\alpha Y) = -\frac{\mathrm{i}n}{a}\hat{P}_1+\hat{W}_{1YY}, \tag{3.1b}$$

$$\mathrm{i}\alpha\hat{U}_1+\hat{V}_{1Y}+\frac{\mathrm{i}n}{a}\hat{W}_1=0. \tag{3.1c}$$

We can eliminate the pressure in the first two of these equations to give an Airy equation whose appropriate solution is

$$\hat{U}_{1Y}+\frac{n}{\alpha a}\hat{W}_{1Y} = B\,\mathrm{Ai}(\xi), \tag{3.2}$$

where

$$\xi = -\frac{(\mathrm{i}\alpha)^{\frac{1}{3}}\Omega}{\alpha}+(\mathrm{i}\alpha)^{\frac{1}{3}}Y. \tag{3.3}$$

Invoking, further, the boundary conditions as $Y \to \infty$ demands

$$(\mathrm{i}\alpha)^{-\frac{1}{3}} B \int_{\xi_0}^{\infty} \mathrm{Ai}\,(\xi)\,\mathrm{d}\xi = \hat{A}_1, \tag{3.4}$$

where

$$\xi_0 = -\frac{(\mathrm{i}\alpha)^{\frac{1}{3}}\Omega}{\alpha}. \tag{3.5}$$

Evaluating (3.1 a, b) on $Y = 0$, and combining yields

$$(\mathrm{i}\alpha)^{\frac{1}{3}} B\,\mathrm{Ai}'\,(\xi_0) = \left(\mathrm{i}\alpha + \frac{\mathrm{i}n^2}{\alpha a^2}\right)\hat{P}_1. \tag{3.6}$$

To complete this solution we require the solution from the upper deck. If we write

$$p_1(\bar{r}) = C^{\frac{1}{4}}\lambda^{\frac{1}{2}}[\hat{p}_1(r)\,E_1 + \mathrm{c.c.}], \quad \bar{r} = C^{\frac{3}{8}}\lambda^{-\frac{5}{4}}(T_\mathrm{w}/T_\infty)^{\frac{3}{2}} r, \tag{3.7}$$

then the function $\hat{p}_1$ which satisfies the pressure displacement law and is finite at infinity is

$$\hat{p}_1(r) = \frac{\mathrm{i}\alpha\hat{A}_1 K_n[\mathrm{i}\alpha r\,(M_\infty^2 - 1)^{\frac{1}{2}}]}{(M_\infty^2 - 1)^{\frac{1}{2}} K_n'[\mathrm{i}\alpha a\,(M_\infty^2 - 1)^{\frac{1}{2}}]} \tag{3.8}$$

(assuming $M_\infty > 1$). Finally, a non-trivial solution to (3.4), (3.6), (3.8) exists only if the following (dispersion) relation is satisfied:

$$\frac{\mathrm{Ai}'(\xi_0)}{\displaystyle\int_{\xi_0}^{\infty} \mathrm{Ai}(\xi)\,\mathrm{d}\xi} = \frac{(\mathrm{i}\alpha)^{\frac{4}{3}}\left[1 + \dfrac{n^2}{\alpha^2 a^2}\right] K_n[\mathrm{i}\alpha a\,(M_\infty^2 - 1)^{\frac{1}{2}}]}{(M_\infty^2 - 1)^{\frac{1}{2}} K_n'\,[\mathrm{i}\alpha a(M_\infty^2 - 1)^{\frac{1}{2}}]} \tag{3.9}$$

Notice that setting $n = 0$ (axisymmetric mode), and defining

$$\alpha = (M_\infty^2 - 1)^{\frac{3}{8}}\bar{\alpha}, \quad \Omega = (M_\infty^2 - 1)^{\frac{1}{4}}\bar{\Omega}, \quad a = (M_\infty^2 - 1)^{-\frac{7}{8}}\hat{a}, \quad \bar{\xi}_0 = -\frac{(\mathrm{i}\bar{\alpha})^{\frac{1}{3}}\bar{\Omega}}{\bar{\alpha}} \tag{3.10}$$

yields the following dispersion relationship:

$$\frac{\mathrm{Ai}'\,(\bar{\xi}_0)}{\displaystyle\int_{\bar{\xi}_0}^{\infty} \mathrm{Ai}\,(\bar{\xi})\,\mathrm{d}\bar{\xi}} = -\frac{(\mathrm{i}\bar{\alpha})^{\frac{4}{3}} K_0[\mathrm{i}\bar{\alpha}\hat{a}]}{K_1[\mathrm{i}\bar{\alpha}\hat{a}]}, \tag{3.11}$$

which is the same dispersion relationship as found by Duck & Hall (1989).

4. The dispersion relationship for $a \ll 1$ and $a \gg 1$

Here we shall discuss the limiting forms of the dispersion relationship in the limits of either very thin or thick cylinders. We recall that axisymmetric modes always occur in pairs and exist for a less than some critical value, with branches having α, Ω tending to infinity when $a \to 0$. Suppose then that $a \to 0$ in (3.9) in such a way that $\alpha a \to 0$. From the series expansions of K_n, K_n' we deduce that

$$\frac{\mathrm{Ai}'\,(\xi_0)}{\chi} \approx (\mathrm{i}\alpha)^{\frac{1}{3}}\frac{n}{a} + \ldots,$$

where

$$\chi = \int_{\xi_0}^{\infty} \mathrm{Ai}\,(s)\,\mathrm{d}s,$$

so that for neutral modes we must have that

$$\xi_0 \approx 2.298\mathrm{i}^{\frac{1}{3}}, \quad \mathrm{Ai}'(\xi_0)/\chi \sim 1.001\mathrm{i}^{\frac{1}{3}},$$

and the neutral values of α, Ω then become

$$\alpha = \left(\frac{1.001a}{n}\right)^3 + \ldots, \quad \Omega = 2.298\left(\frac{1.001a}{n}\right)^2 + \ldots. \tag{4.1a,b}$$

Thus, unlike the axisymmetric problem, there are neutral solutions for $a \ll 1$ which have $\alpha, \Omega \to 0$. This is a result of some significance because it means that for thin cylinders non-axisymmetric modes are the most important since they will occur at lower values of the Reynolds number than do the axisymmetric modes.

However, at finite Reynolds numbers, where our asymptotic approach is not valid, we cannot say which type of mode is the most dangerous. Indeed, as is the case for the incompressible problem at finite Reynolds numbers, it is then not clear that the concept of a mode or critical Reynolds number is even tenable.

In fact there is another asymptotic solution of (3.9) available in the limit $a \ll 1$. The second mode again has $\alpha a \ll 1$ but $|\xi_0|$ now tends to infinity. For large values of $|\xi_0|$ we can replace $\mathrm{Ai}'(\xi_0)/\chi$ by $-\xi_0\{1-\xi_0^{-\frac{3}{2}}+\ldots\}$ and we again approximate $K_n(z)/K_n'(z)$ using the series expansion of the modified Bessel function. After equating the dominant real and imaginary parts of (3.9) we then deduce that

$$\alpha = \left(\frac{2^{n+\frac{1}{2}}n!\,n-1!}{\pi(M_\infty^2-1)^n}\right)^{\frac{1}{2n+\frac{1}{2}}} a^{-\left(\frac{4n-3}{4n+1}\right)} + \ldots, \quad \Omega = \frac{n\alpha}{\alpha} + \ldots. \tag{4.2a,b}$$

Thus $\alpha \sim a^{-\left(\frac{4n-3}{4n+1}\right)}$ in this limit and since $(4n-3)/(4n+1)$ increases monotonically from $\frac{1}{5}$ to 1 when $n \to \infty$ it follows that the largest values of α correspond to n tending to infinity. It follows that the distances between the upper and lower branches of (3.9) for $a \to 0$ increase with n. Thus the band of unstable wavenumbers increases with the azimuthal wavenumber n; this does not of course necessarily mean that the maximum growth rate will occur for $n \gg 1$. We shall return to the latter point in the next section.

Now we consider the structure of (3.9) in the limit $a \to \infty$. First we note that if we let $a \to \infty$ with n fixed then, if $|\alpha a| \gg 1$, (3.9) reduces to

$$\frac{\mathrm{Ai}'(\xi_0)}{\chi} = -\frac{(\mathrm{i}\alpha^{\frac{4}{3}})}{(M_\infty^2-1)^{\frac{1}{2}}} + \ldots,$$

which has no neutral solutions. Thus we must instead consider the double limit $n \to \infty$, $a \to \infty$ but with n/a held fixed. This means that the wavelength in the aximuthal direction is comparable with that in the streamwise direction. The asymptotic form for a modified Bessel function of large argument and order yields

$$\frac{K_n(\mathrm{i}\alpha a\,(M_\infty^2-1)^{\frac{1}{2}})}{K_n'(\mathrm{i}\alpha a(M_\infty^2-1)^{\frac{1}{2}})} = \frac{-\mathrm{i}\alpha a\,(M_\infty^2-1)^{\frac{1}{2}}}{n\left[1-\dfrac{\alpha^2a^2}{n^2}(M_\infty^2-1)\right]^{\frac{1}{2}}} + \ldots,$$

where we have assumed that $(\alpha^2a^2/n^2)(M_\infty^2-1) < 1$. It follows that neutral solutions of (3.9) again occur when $\xi_0 \approx -2.298\mathrm{i}^{\frac{1}{3}}$ so that

$$1.001\left[1-\frac{\alpha^2a^2}{n^2}(M_\infty^2-1)\right]^{\frac{1}{2}} = \frac{\alpha^{\frac{7}{3}}a}{n}\left(1+\frac{n^2}{\alpha^2a^2}\right) + \ldots,$$

or if we write $\beta = n/a$ we obtain

$$1.001\left[1-\frac{\alpha^2}{\beta^2}(M_\infty^2-1)\right]^{\frac{1}{2}} = \alpha^{\frac{7}{3}}\beta^{-1}\left(1+\frac{\beta^2}{\alpha^2}\right)+\ldots, \tag{4.3}$$

which is the supersonic planar neutral dispersion relationship of Smith (1989).

It is an easy matter to show that (3.9) has no neutral solutions for $n, a \gg 1$ with $(\alpha^2 a^2/n^2)(M_\infty^2-1) > 1$. Thus neutral disturbances exist only for aximuthal wavenumbers satisfying

$$n^2 > \alpha^2 a^2 (M_\infty^2 - 1). \tag{4.4}$$

We shall discuss the above asymptotic results after describing our numerical results for the dispersion relationship (3.9).

5. Results and discussion

The dispersion relationship (3.9) was solved for the neutral state ($\mathrm{Im}\{\alpha\} = \mathrm{Im}\{\Omega\} = 0$), for prescribed M_∞, n, using a straightforward Newton iterative scheme. Results for $M_\infty = \surd 2$ are shown in figures 1(*a*) (Ω–a curve), 1(*b*) (α–a curve); for $M_\infty = 5$ in figures 2(*a*) (Ω–a curve), 2(*b*) (α–a curve); for $M_\infty = 10$ in figures 3(*a*) (Ω–a curve), 3(*b*) (α–a curve).

As with the axisymmetric case considered by Duck & Hall (1989), for fixed a less than a_c, (the critical body radius above which the flow is stable) two possible modes exist. However, there is an important difference between this and the axisymmetric case, namely the behaviour of the 'lower branch,' which here has $\Omega \to 0$, $\alpha \to 0$ as $a \to 0$, whilst in the axisymmetric case it is found that $\Omega \to \infty$, $\alpha \to \infty$ as $a \to 0$.

Further observations regarding these $n \neq 0$ results are

1. as n increases, a_c increases (indeed, all the a_c determined here were greater than the a_c obtained in the $n = 0$ case);
2. as M_∞ increases, for n fixed, a_c decreases;
3. kinks were consistently observed on the upper branch of the $\alpha - a$ curve for the smaller value of n.

In figure 1, we have also shown (as broken curves) the lower and upper branches predicted by the asymptotic analysis of §4. We see that the asymptotic theory accurately predicts the neutral wavenumbers and frequencies over a wide range of values of a. A similar agreement is found for the higher Mach number cases but is not shown in the figures.

Now let us turn to a matter of some practical importance; namely the question of which mode is the most important in any given practical situation. There are two obvious ways to classify the 'most dangerous' mode. First the high Reynolds number form of the neutral curve can be used to predict which mode becomes linearly unstable at the lowest value of the Reynolds number with the disturbance streamwise wavenumber held fixed. Figure 1(*a, b*) shows that at a given value of a there is an infinite sequence of neutral values of α. The neutral value of α decreases with n so that at high Reynolds number the lower branch of the neutral curve for the mode with wavenumber N_1 is to the left of that with wavenumber N_2 if $N_1 > N_2$. Thus, as with incompressible plane flows, this classification of the 'most dangerous' mode merely suggests that the importance of a mode increases with its azimuthal wavenumber. However, this classification takes no account of the different growth rates of the modes. Thus an alternative definition of the 'most dangerous' mode at any values of a and M_∞ is simply that with the largest growth rate. With the latter

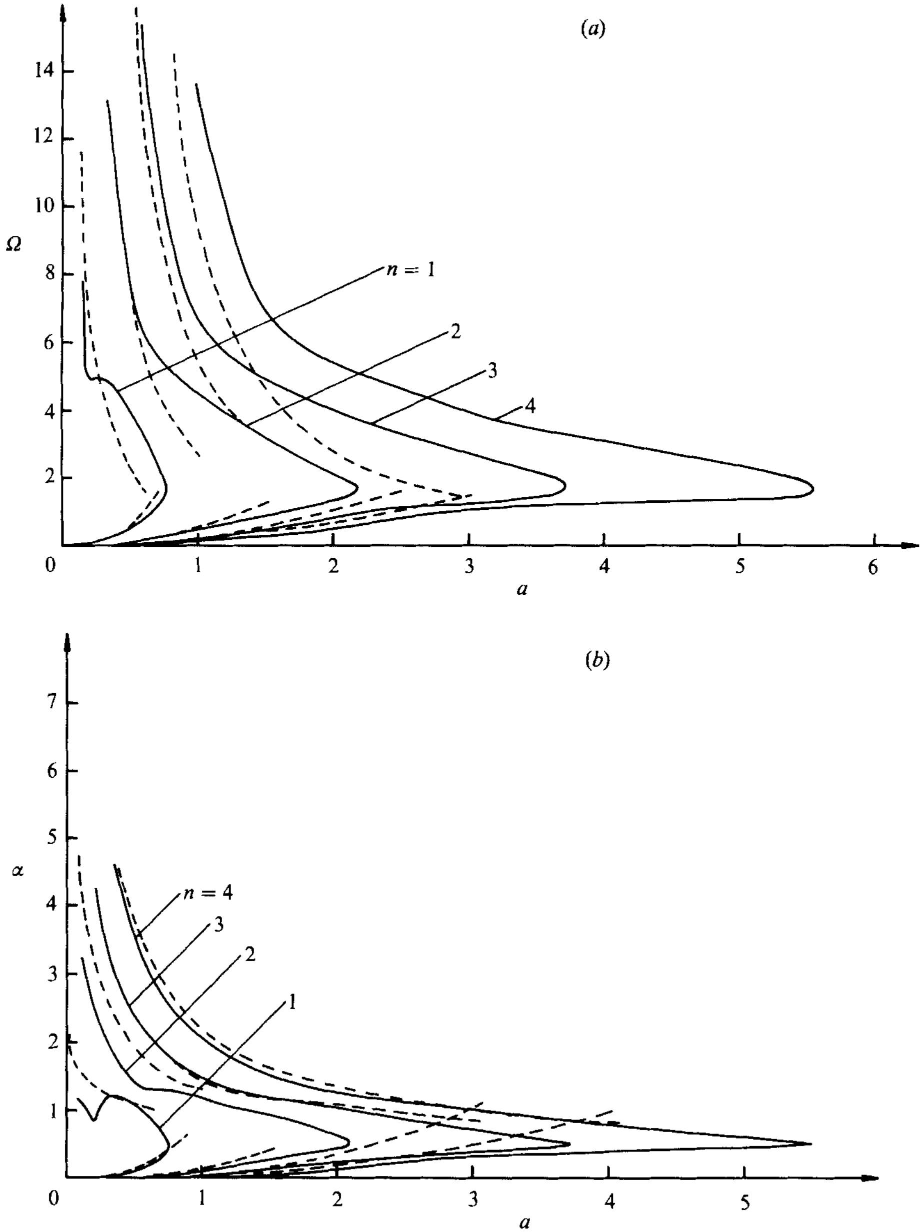
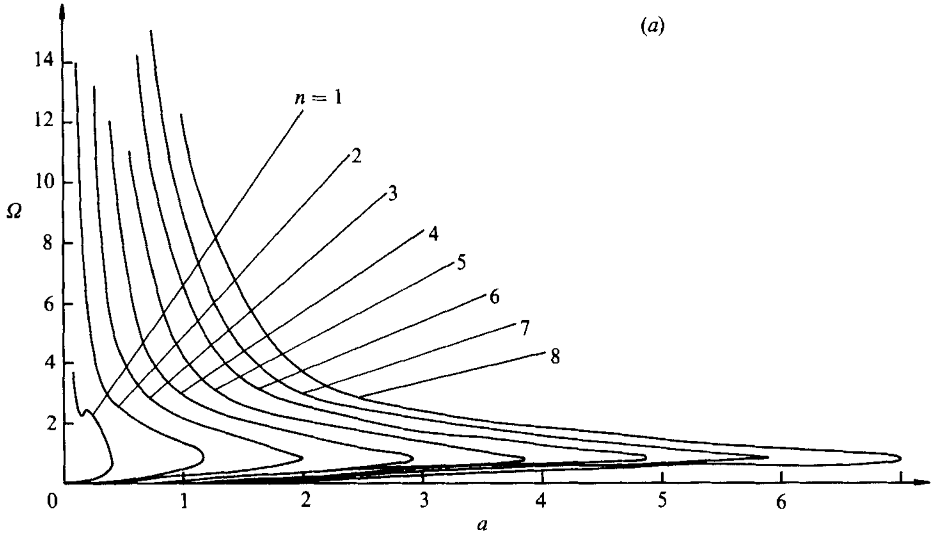


FIGURE 1. (a) Neutral Ω–a curve, (b) neutral α–a curve; $M_\infty = \sqrt{2}$.

notion in mind we have shown in figure 4 the results of some non-neutral calculations for the case $M_\infty = \sqrt{2}$ and $a = 2$.

The frequency Ω is taken to be real and when the corresponding complex value of the wavenumber α is calculated we see that the curves are unstable at frequencies between the two neutral values. Furthermore, the growth rates initially increase with n so that, at least at the values of a and M_∞ chosen, the non-axisymmetric modes become progressively more unstable with increasing n. In fact the most dangerous mode, i.e. the one with the largest growth rate, occurs for $n = 6$ for $a = 2$, $M_\infty = \sqrt{2}$.



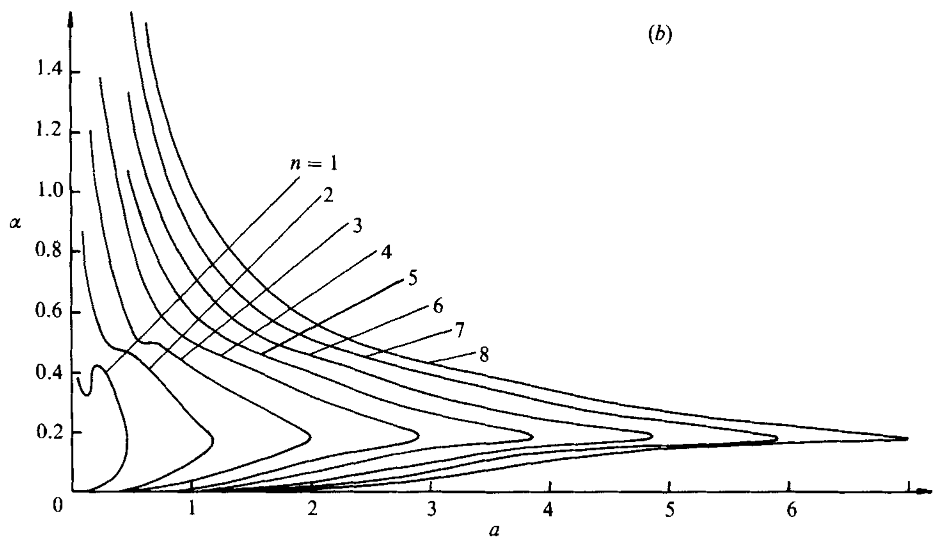


FIGURE 2. (*a*) Neutral Ω–a curve, (*b*) neutral α–a curve; $M_\infty = 5$.

Similar calculations were performed at a finite number of points in the (a, M_∞)-plane. We were unable to perform calculations at a sufficiently larger number of points for us to identify the most dangerous mode everywhere. However, our calculations were sufficient to convince us of the following statements regarding the relative importance of the modes:

1. for a given value of a the mode number of the most dangerous mode increases with M_∞;

2. the axisymmetric mode always has a smaller growth rate than one of the non-axisymmetric modes.

Unfortunately there are no numerical results available for Tollmien–Schlichting waves in the configuration we have discussed. It is clear from our analysis that any numerical investigation should allow for the possibility of non-axisymmetric modes. Fortunately, unlike the planar case, only integer values of the wavenumber in the

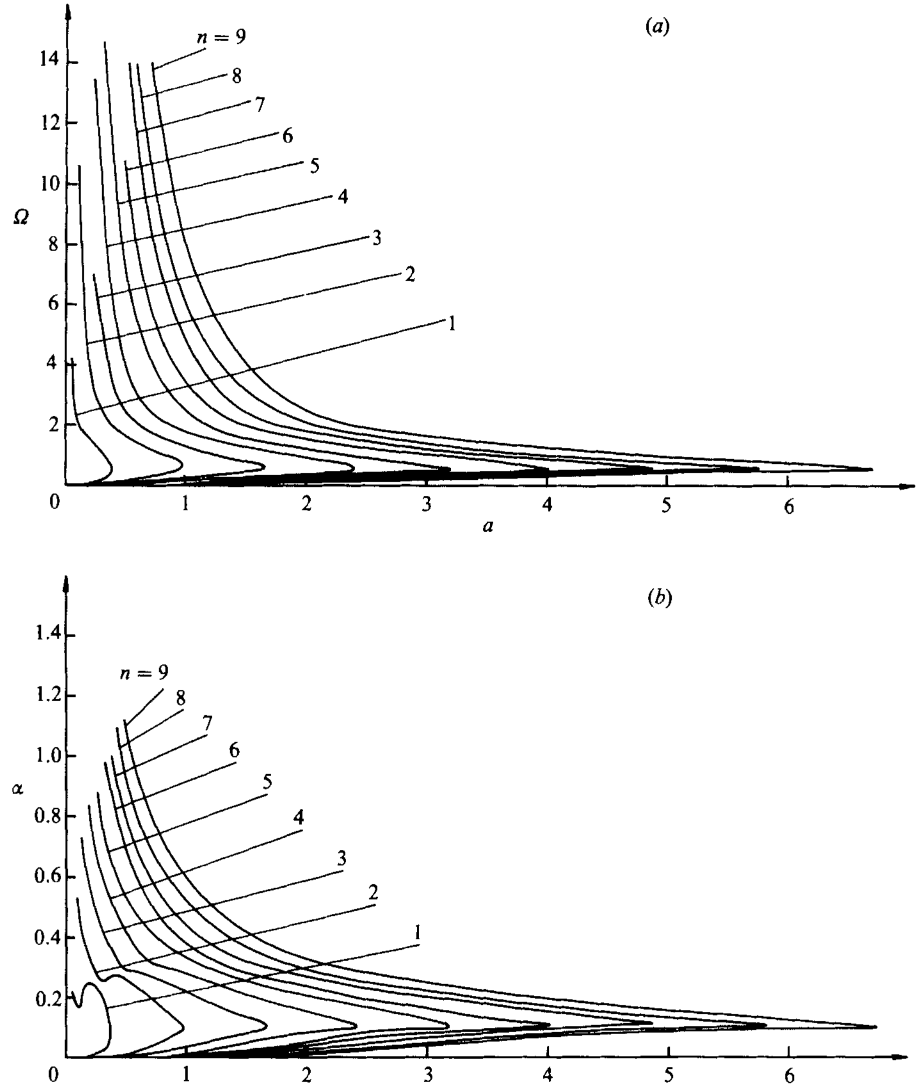


FIGURE 3. (*a*) Neutral Ω–a curve, (*b*) neutral α–a curve; $M_\infty = 10$.

cross-stream direction are physically acceptable so there is a selection mechanism to reduce the number of modes which must be considered in any numerical investigation. Our calculations suggest that experimentally it is almost certainly non-axisymmetric Tollmien–Schlichting waves that will be stimulated by an input disturbance of frequency appropriate to the triple-deck timescale. The question of which mode 'wins' the competition set up when the modes interact can only be answered by a nonlinear theory. Moreover the competition set up when other disturbances such as inviscid disturbances and Görtler vortices exist is again beyond the scope of the present calculation.

The authors acknowledge the helpful comments made by the referees on the original form of this paper. This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the authors

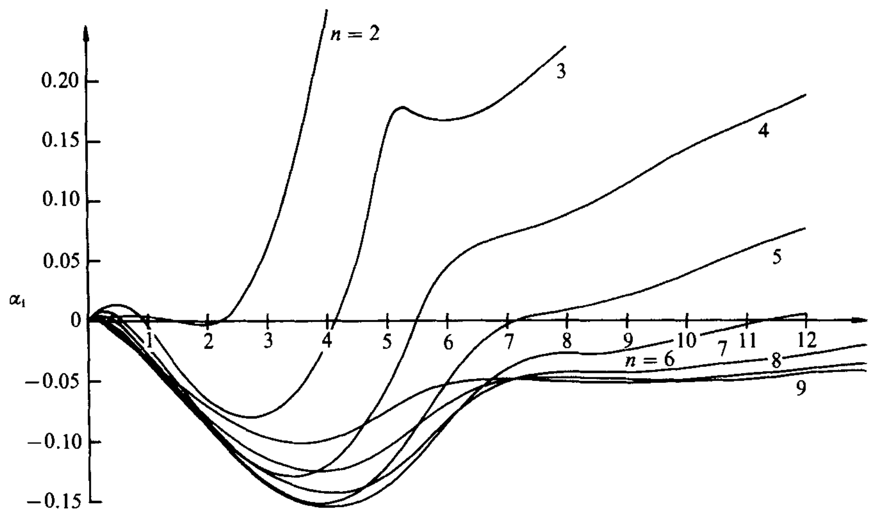


FIGURE 4. Spatial growth rates (α_i), Ω real, $M_\infty = \surd 2$, $a = 2$, n as indicated.

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